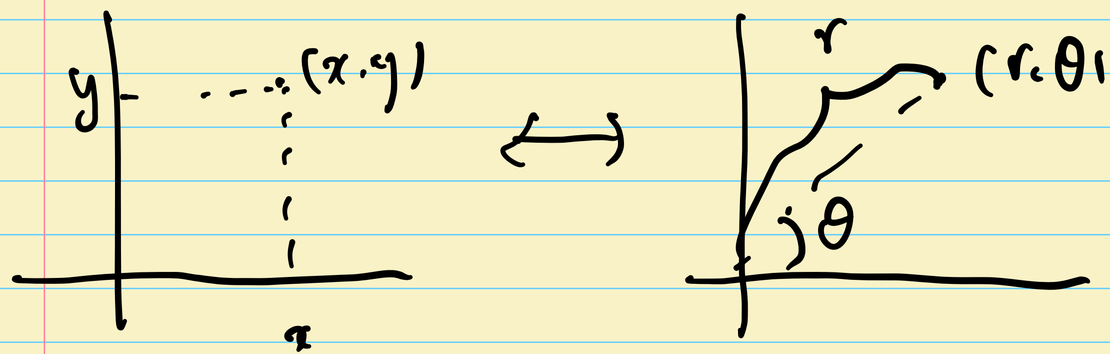
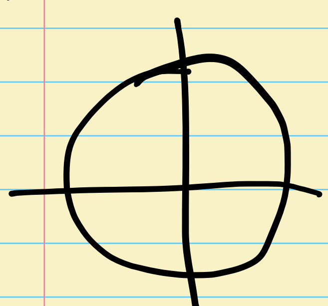
Geometric objects in $\mathbb{R}^n$

curve. $\vec{x}: I \to \mathbb{R}^n$, $\vec{x}'(t)$, arclength $= \int_a^b \|\vec{x}'(t)\|\,dt$

Polar coordinates — suitable for some geometric objects.

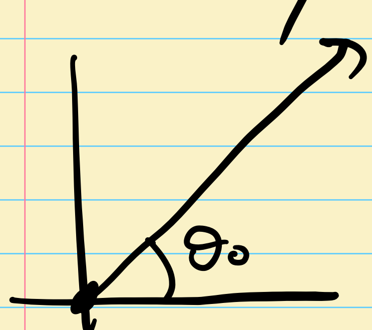


Example circle of radius r_0 centered at origin.



	xy-coordinate	polar coordinate
eqn	$x^2+y^2=r_0^2$	$r=r_0$
parametrization	$(x,y) = (r_0\cos t, r_0\sin t)$, $t\in[0,2\pi]$	$(r,\theta) = (r,t)$, $t\in[0,2\pi)$.

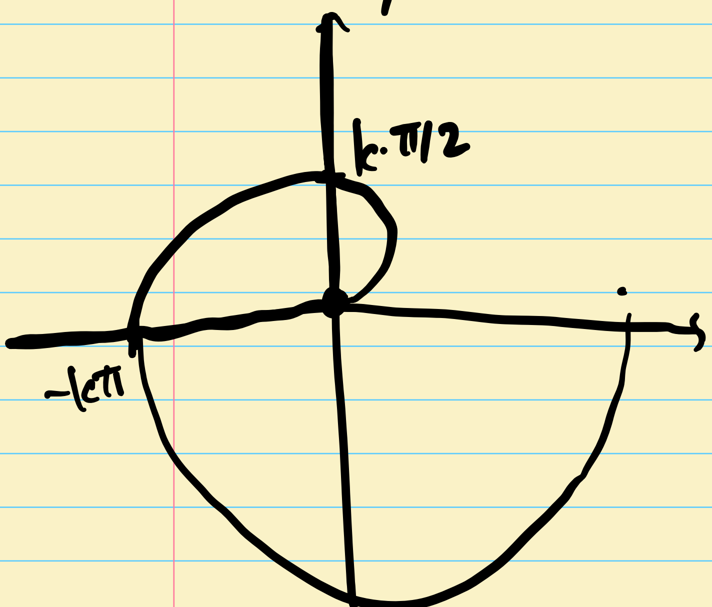
Example half-ray from the origin



	xy coord	polar
parametrization	$(x,y) = (t, \tan\theta_0 \cdot t)$, $t\in[0,\infty)$	$(r,\theta) = (t,\theta_0)$, $t\in[0,\infty)$
equation	?	$\theta=\theta_0$ (Recall. our convention $r\geq 0$)

Example Archimedes spiral

$k>0$ a constant. By definition, it is given by $r = k\theta$ in polar coordinate



$(r,\theta) = (kt, t),\ t \in [0,\infty)$

in parametrization

In xy-coordinate

- parametrization

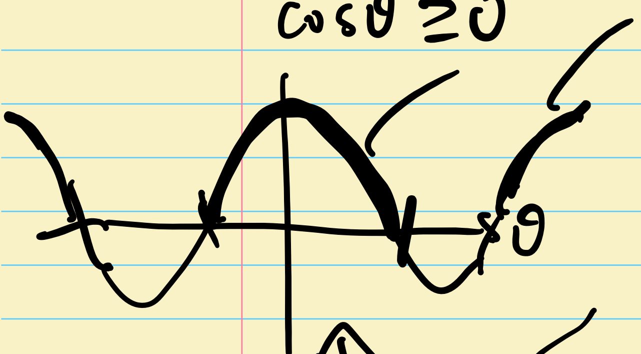
$(x,y) = (kt\cos t,\ kt\sin t)$

- equation ?

Example $r = 4\cos\theta$

Note that we assumed $r \geq 0$; $4\cos\theta \geq 0$

$\cos\theta \geq 0$

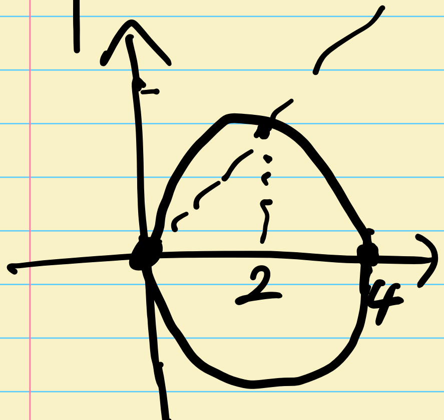


$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\theta = 0$; $r = 4$

$\theta = \pi/4$; $r = 4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$

$\theta = \pi/2$; $r = 0$



In fact it is a circle. why?

$r = 4\cos\theta$

$\Rightarrow r^2 = 4r\cos\theta$

$\Rightarrow x^2+y^2 = 4x$

$\Rightarrow (x-2)^2+y^2 = 2^2$

: the circle of radius 2 centered at $(2,0)$.

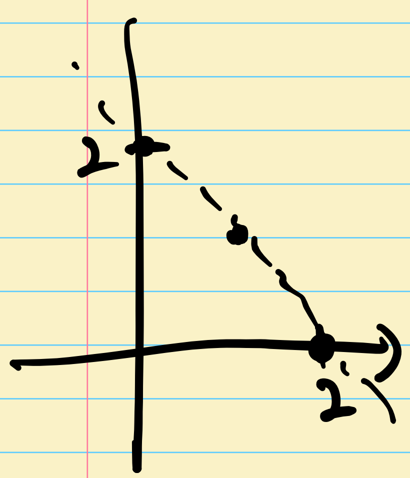
If we let $x = r\cos\theta = 4\cos^2\theta$

$y = r\sin\theta = 4\sin\theta\cos\theta$

Example $r\cos\left(\theta-\frac{\pi}{4}\right) = \sqrt{2}$

$\theta = 0$; $r\cos\left(-\frac{\pi}{4}\right) = \sqrt{2} \Rightarrow r = 2$

$\theta = \pi/2 \Rightarrow r = 2$, $\theta = \pi/4 \Rightarrow r = \sqrt{2}$



In fact it is a line

$$\sqrt{2} = r\cos\left(\theta-\frac{\pi}{4}\right)$$

$$= r\left(\cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}r\cos\theta + \frac{1}{\sqrt{2}}r\sin\theta$$

$$= \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y$$

$\therefore x+y = 2$.

Rmk our convention : $r \in [0, \infty)$

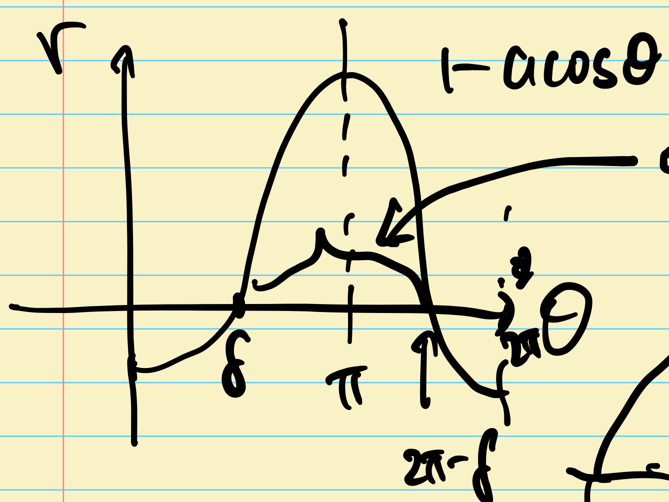
What happens we allow $r \in (-\infty, \infty)$?

eg $r = 1 - a\cos\theta$ $\quad (a > 1$ constant$)$

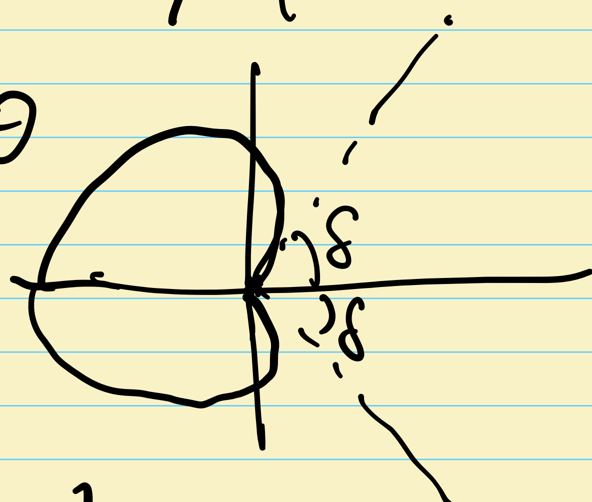
i) If $r \in [0, \infty)$

$1 - a\cos\theta$ must be ≥ 0.

$\Rightarrow \cos\theta \leq \frac{1}{a}$

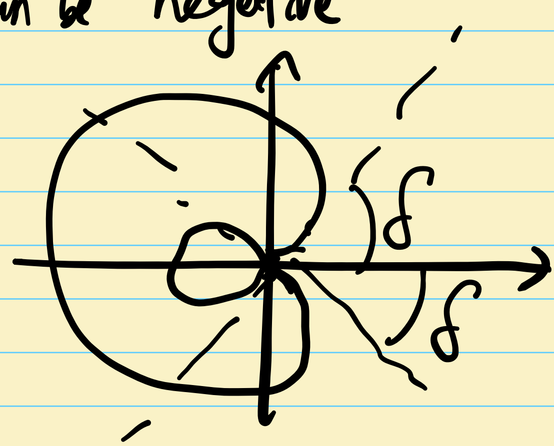


only possible θ.



ii) If $r \in (-\infty, \infty)$

$1 - a\cos\theta$ can be negative

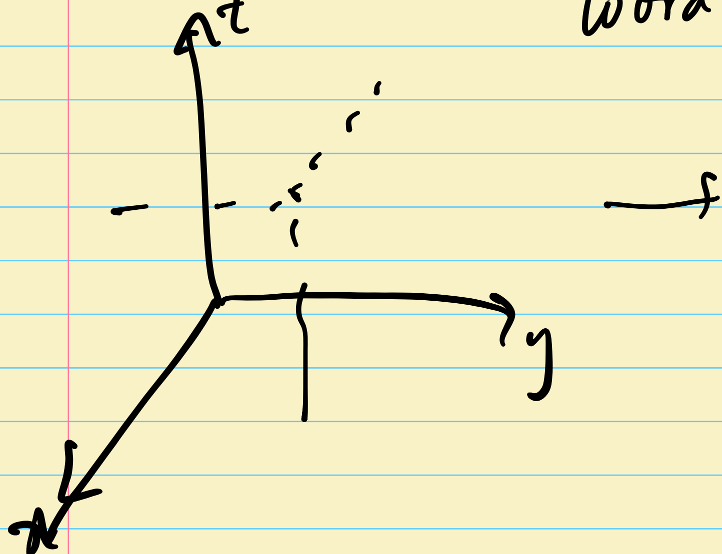


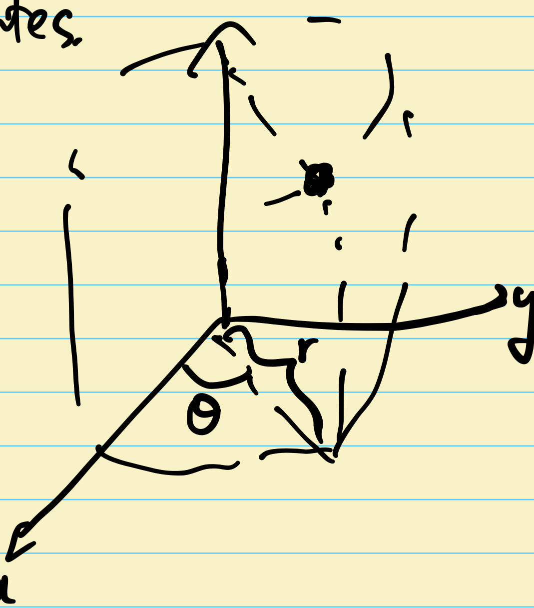
Other coordinates systems in $\mathbb{R}^3$

Cylindrical coordinates

$$(x, y, z) \longrightarrow (r, \theta, z)$$

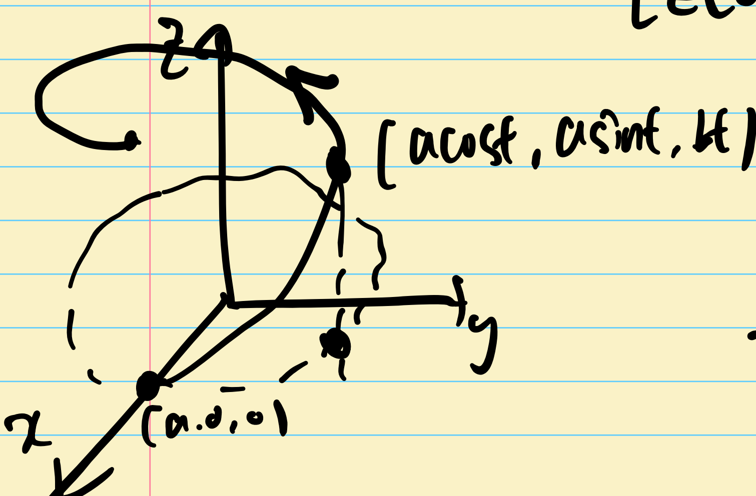
Express x, y using polar coordinates.





Given (r, θ, z), $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

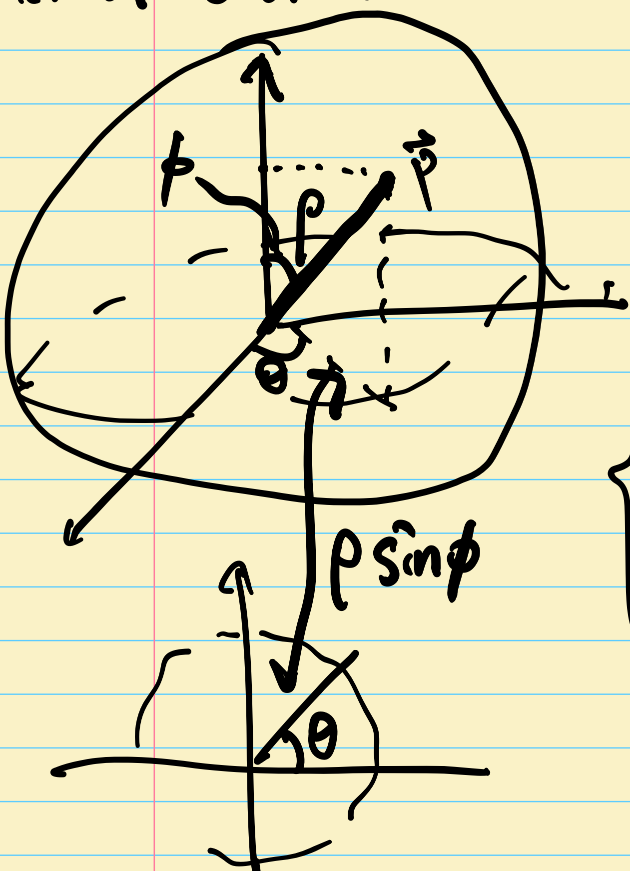
<u>Example</u> Recall helix : $\vec{x}(t) = (a\cos t, a\sin t, bt)$
$t \in [0, 2\pi]$.



Using cylindrical coordinate,

$$\begin{cases} r = \sqrt{(a\cos t)^2 + (a\sin t)^2} = a \\ \theta = t, \quad z = bt \end{cases}$$

Spherical coordinates



Describe a point $P \in \mathbb{R}^3$ by

ρ = distance from the origin

$= \sqrt{x^2+y^2+z^2}$

$\theta = \theta$ as in cylindrical coordinates.

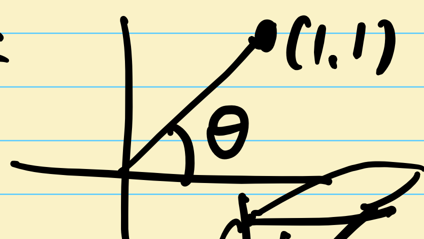
ϕ = angle from positive z-axis to $\vec{OP}$

$(\phi \in [0, \pi])$

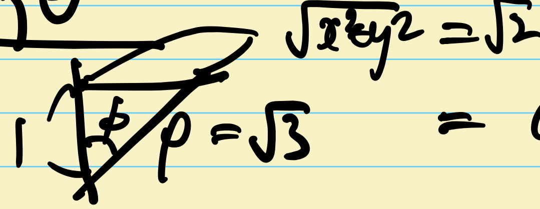
$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

<u>Example</u> A point $(1,1,1) \in \mathbb{R}^3$. in polar coordinates,

$$\rho = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$\theta =$ 

$= \frac{\pi}{4}$

$\phi =$ 

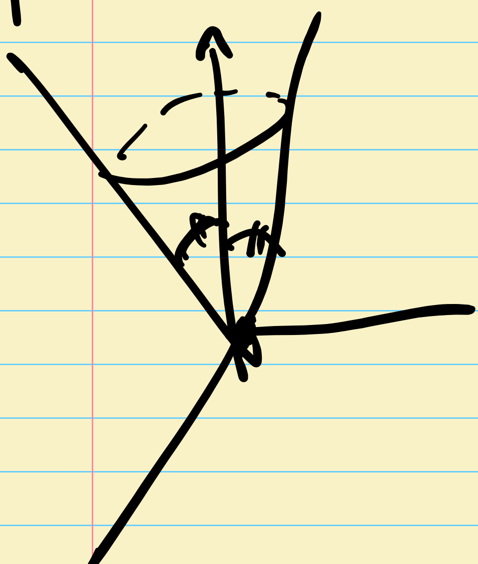
$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Example A sphere of radius 2 centered at origin

xyz-coord $\qquad\qquad$ spherical coord

$x^2+y^2+z^2=2^2$ $\qquad\qquad$ $\rho=2.$

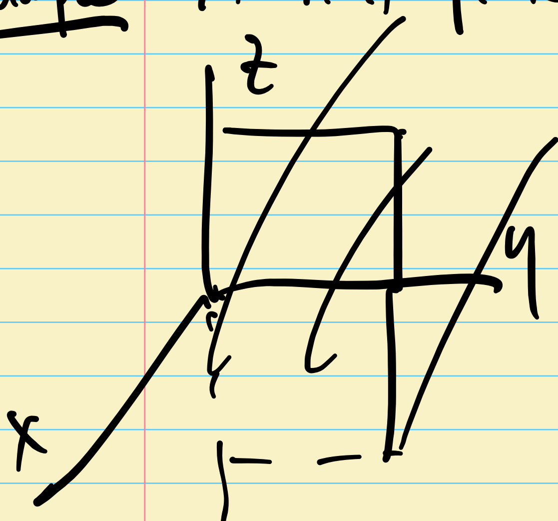
Example A cone : In spherical coordinates,



$$\phi=\frac{\pi}{4}$$

In xyz-coordinate ?

Example A half plane

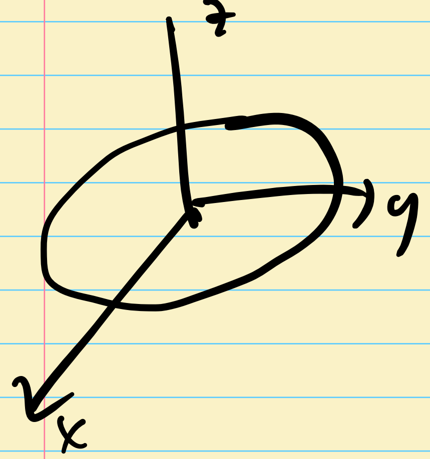


$$\begin{cases} x=0 \\ y\geq 0 \end{cases}$$

In spherical coordinate,

$\theta=0.$

Example A circle on xy-plane, radius 3 centered at origin.



xyz-coord

$$\begin{cases} z=0 \\ x^2+y^2=3^2 \end{cases}$$

or

spherical coord

$$\begin{cases} \rho=3 \\ \phi=\pi/2 \end{cases}$$

or parametrically

$$\begin{cases} \rho=3 \\ \theta=t \quad , \quad t\in[0,2\pi] \\ \phi=\pi/2 \end{cases}$$

(topological) terminology in $\mathbb{R}^n$

Let $\vec{x}_0 \in \mathbb{R}^n$, $\varepsilon>0$. Define

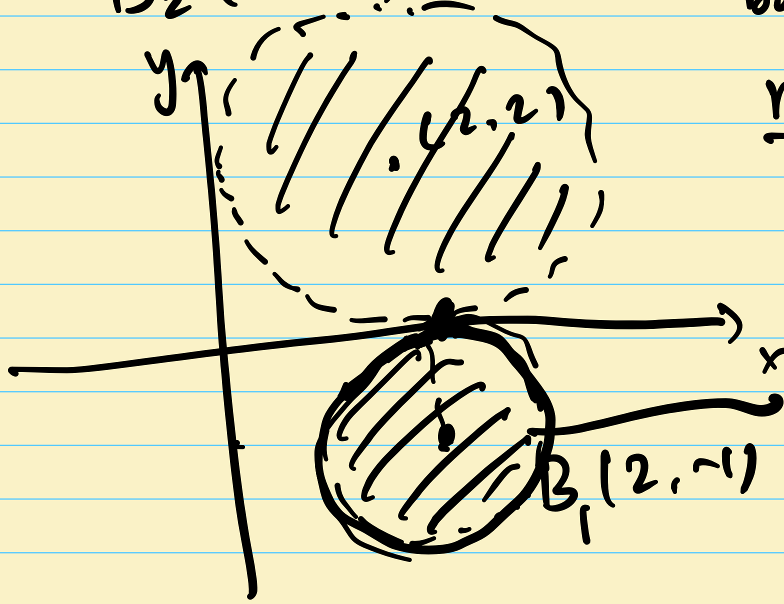
$$B_\varepsilon(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}-\vec{x}_0\| < \varepsilon \}$$

called an <u>open</u> ball with radius ε centered at $\vec{x}_0$.

$$\overline{B_\varepsilon(\vec{x}_0)} = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}-\vec{x}_0\| \le \varepsilon \}$$

called a closed ball with radius ε centered at $\vec{x}_0$.

eg $B_2(2,2)$ in $\mathbb{R}^2$



boundary circle not included

(2,0)

boundary circle included

Def Let S be a subset of $\mathbb{R}^n$

① The interior of S is the set

$$\text{Int}(S) = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{For some } \varepsilon > 0, \\ B_\varepsilon(\vec{x}) \subset S \end{array} \right\}$$

Points in $\text{Int}(S)$ are called interior points of S

② The exterior of S is the set

$$\text{Ext}(S) = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{For some } \varepsilon > 0, \\ B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \setminus S \end{array} \right\}$$

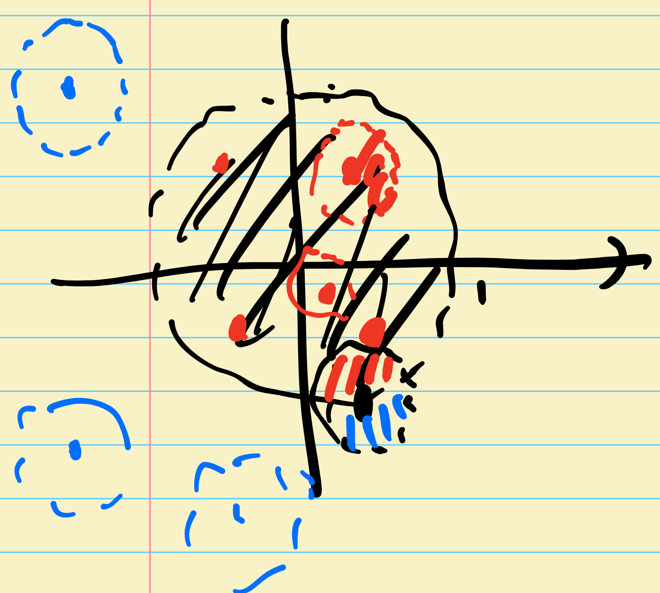
Points in $\text{Ext}(S)$ are called exterior points of S.

③ The boundary of S is the set

$$\partial S = \left\{ \vec{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \text{For any } \varepsilon > 0, \\ B_\varepsilon(x) \cap S \neq \phi \\ B_\varepsilon(x) \cap (\mathbb{R}^n \setminus S) \neq \phi \end{array} \right\}$$

Points in ∂S are called the boundary points of S.

Example $S = B_1(\vec{0}) \subset \mathbb{R}^2$



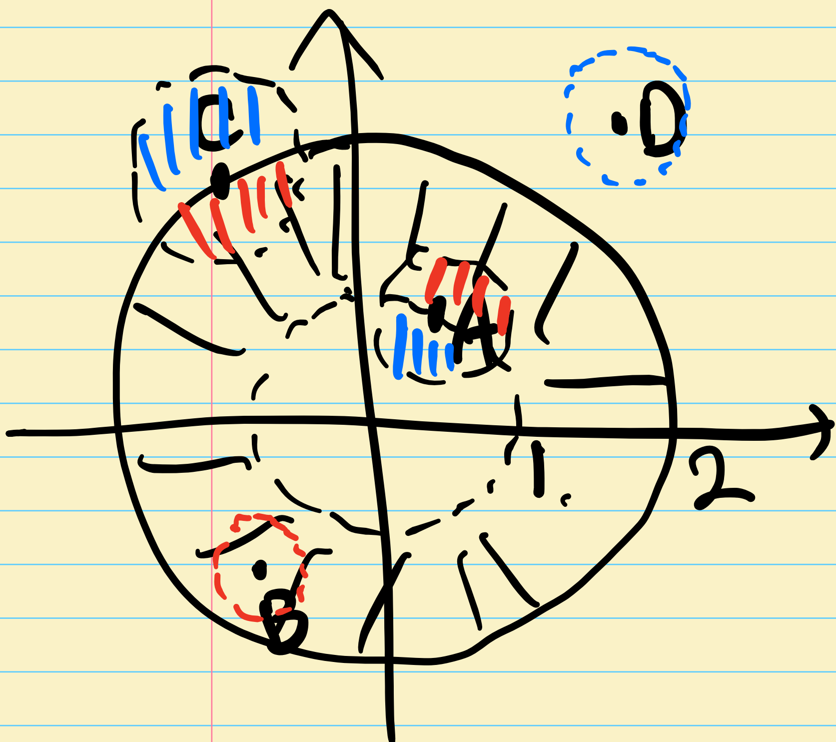
$\mathrm{Int}(S) = S$

$= \{ x^2+y^2 < 1^2 \}$

$\mathrm{Ext}(S) = \{ x^2+y^2 > 1^2 \}$

$\partial S = \{ x^2+y^2 = 1 \}$

Example $S = \{(x,y) \in \mathbb{R}^2 \mid 1^2 < x^2+y^2 \le 2^2\}$



$A, D \notin S$

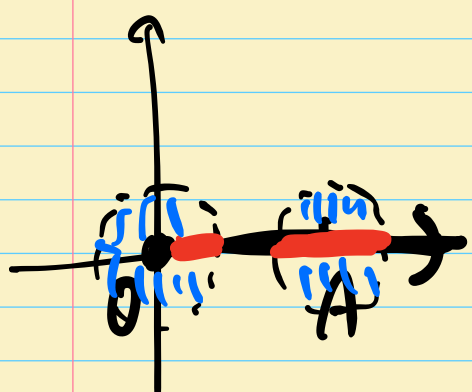
$B, C \in S$.

$\mathrm{Int}\, S \ni B$

$\mathrm{Ext}\, S \ni D$

$\partial S \ni A, C$

Example $S =$ non-negative x-axis in $\mathbb{R}^2$

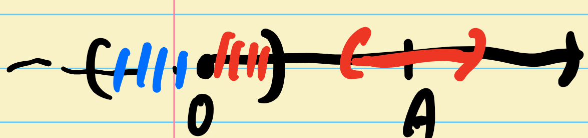


$0, A \in S$

$0, A \in \partial S$

$\mathrm{Int}(S) = \phi$.

Example $S =$ non-negative real numbers in $\mathbb{R}^1$



$0, A \in S$

$A \in \mathrm{Int}(S)$

$0 \in \partial S$

Prop Let $S \subseteq \mathbb{R}^n$

① $\mathbb{R}^n$ is the disjoint union of $Int(S)$, $Ext(S)$, ∂S.

② $Int(S) \subseteq S$, $Ext(S) \subseteq \mathbb{R}^n \setminus S$

A point in ∂S may or may not be in S.

Def A subset $S \subseteq \mathbb{R}^n$ is called

① open if $\forall x \in S$, there exist $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.

② closed if $\mathbb{R}^n \setminus S$ is open.

Equivalent definition A subset $S \subseteq \mathbb{R}^n$ is

① open if $S = Int(S)$

② closed if $S = Int(S) \cup \partial S$.

Exercise open ball $B_\varepsilon(\vec{x})$ is open.

closed " $\overline{B_\varepsilon(\vec{x})}$ " closed.

Subset $S \subseteq \mathbb{R}^2$	$B_1(\vec{0})$	$\overline{B_1(\vec{0})}$	S^1	$\mathbb{R}^2$	ϕ
$\mathrm{Int}(S)$	$B_1(\vec{0})$	$B_1(\vec{0})$	ϕ	$\mathbb{R}^2$	ϕ
$\mathrm{Ext}(S)$	$\{x^2+y^2>1\}$	$\{x^2+y^2>1\}$	$\mathbb{R}^2 \setminus S^1$	ϕ	$\mathbb{R}^2$
∂S	$S^1=\{x^2+y^2=1\}$	S^1	S^1	ϕ	ϕ
open?	Yes	No	No	Yes	Yes
closed?	No	Yes	Yes	Yes	Yes

<u>Rmk</u>. The subsets of $\mathbb{R}^n$ which are both open and closed: $\mathbb{R}^n$ and ϕ.

- Some subsets of $\mathbb{R}^n$ are neither open nor closed.

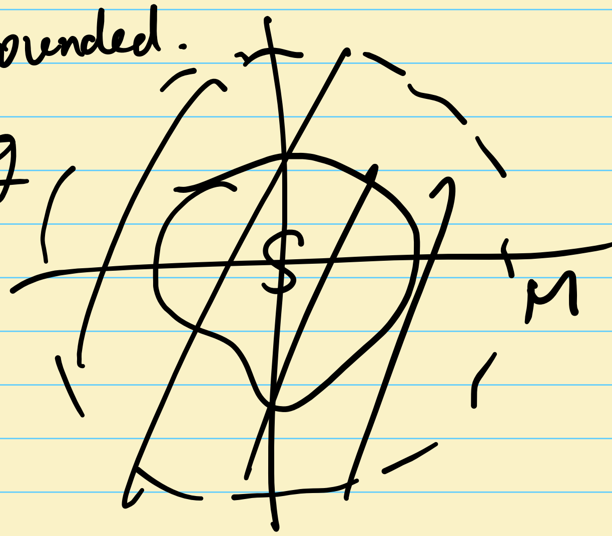
eg. $\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 \leq 4\}$ is not open not closed

- $[0,1) \subseteq \mathbb{R}$
- $\mathbb{Q} \subseteq \mathbb{R}^1$

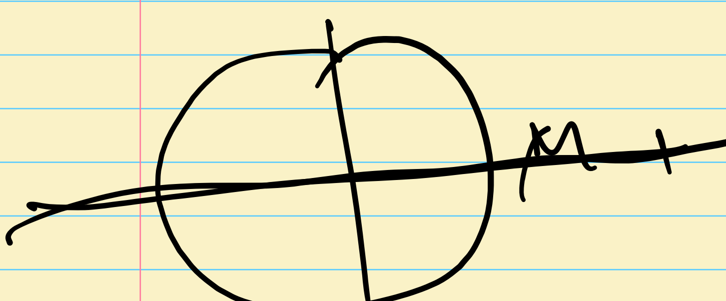
· For any $S \subseteq \mathbb{R}^n$,
$\text{Int}(S)$, $\text{Ext}(S)$ are open in $\mathbb{R}^n$.
∂S is closed in $\mathbb{R}^n$.

<u>Def</u> Let $S \subseteq \mathbb{R}^n$ a subset.
① S is called <u>bounded</u> if $\exists M > 0$ s.t.
$S \subseteq B_M(\vec{0})$
S is called <u>unbounded</u> if S is not bounded.

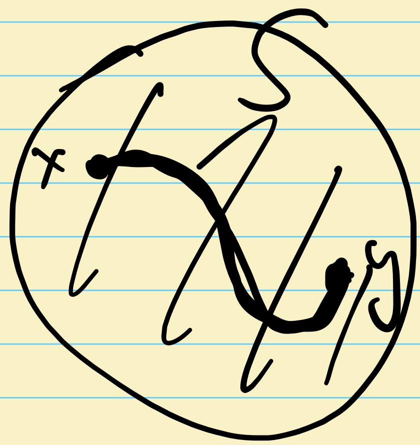
<u>eg</u>



· $B_1(0,0)$, $\overline{B_1(0,0)}$ are bounded in $\mathbb{R}^2$
· $\mathbb{R}^2$ itself is unbounded
· x-axis is unbounded

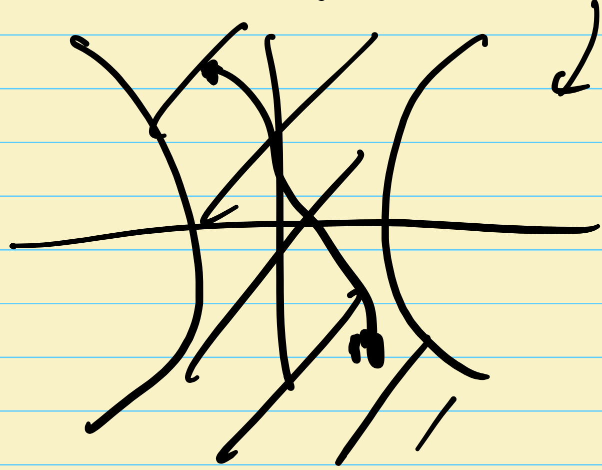


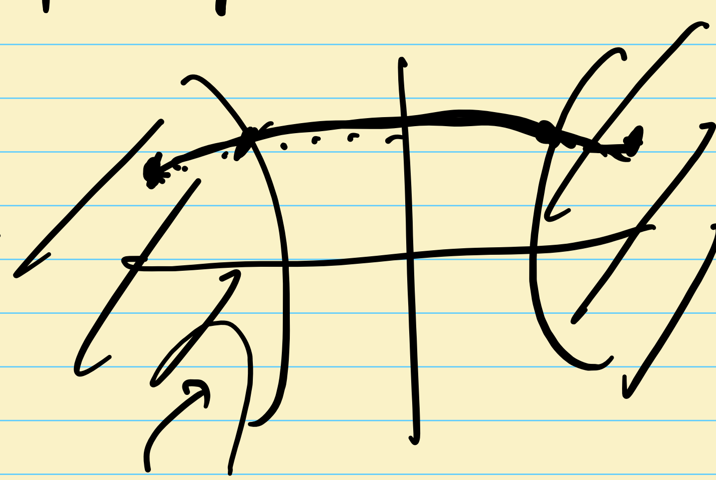
② S is called <u>path-connected</u> if any two points in S can be connected by a curve in S.



eg · $\mathbb{R}^2$ is path-connected

· $\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \leq 1\}$ is path connected





· $\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \geq 1\}$ is <u>not</u> path connected

Thm (Jordan curve theorem)

A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two path-connected components, with one bounded and the other unbounded.

eg